

# Stationary random walks on the lattice

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## Abstract

We consider translation invariant measures on configurations of nearest-neighbor arrows on the integer lattice. Following the arrows from each point on the lattice produces a family of semi-infinite non-crossing walks. We classify the collective behavior of these walks under mild assumptions: they either coalesce almost surely or form bi-infinite trajectories. Bi-infinite trajectories form measure-preserving dynamical systems, have a common asymptotic direction in 2d, and possess other nice properties. We use our theory to classify the behavior of non-crossing semi-infinite geodesics in stationary first- and last-passage percolation. We also partially answer a question raised by C. Hoffman about the limiting empirical measure of weights seen by geodesics.

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## 1 Introduction

Let  $(\Omega, \mathcal{F}, \mathbb{P}, \{T^z\}_{z \in \mathbb{Z}^d})$  be a  $\mathbb{Z}^d$  measure-preserving dynamical system. Consider a family of measurable walks on the lattice  $\{X_z\}_{z \in \mathbb{Z}^d}$ , where each  $X_z: \Omega \times \mathbb{N} \rightarrow$

$\mathbb{Z}^d$  is a nearest-neighbor path that starts at  $z$ . We assume that these walks have been created in a stationary (translation-invariant) way; i.e., almost surely,

$$X_z(\omega, k) = X_0(T^z\omega, k) \quad \forall k \in \mathbb{N}.$$

This implies that if two walks meet at a point at some time, then they remain together in the future; i.e., the two walks must coalesce and *cannot cross each other*. Because of this coalescence, we may assume that there is a stationary vector-field  $\alpha$  that is the discrete time-derivative of the walks:

$$\alpha(\omega, z) = X_z(\omega, 1) - X_z(\omega, 0). \quad (1)$$

The  $\alpha$  function takes values in  $\mathbf{A} \subset \{\pm e_1, \dots, \pm e_d\}$ , and we call a particular  $\alpha$  value an *arrow*. We generally study two cases: one where the walks satisfy a mild “line-crossing” assumption, and there is no restriction on  $\mathbf{A}$ , and the second with *directed* walks, where  $\mathbf{A} = \{e_1, \dots, e_d\}$ . One might think of the walks as the flow generated by the stationary vector field of arrows. We call an arrow configuration *non-trivial* if  $\alpha$  is not a constant almost surely. The canonical walk  $X(\omega)$  starts at the origin and  $\alpha(\omega)$  is its derivative at time 0. We’ll omit the  $\omega$  from the notation when it is clear from context.

We frequently speak of configurations on the lattice: for any  $\omega$ , this refers to the collection of walks  $\{X_z(\omega)\}_{z \in \mathbb{Z}^d}$  or equivalently, the collection of directions  $\{\alpha(T^z\omega)\}_{z \in \mathbb{Z}^d}$ .

As a first example, consider independent and identically distributed (iid) arrows in  $\mathbf{A} = \{e_1, e_2\}$  (with probabilities  $p$  and  $1 - p$ ) on each point of the lattice  $\mathbb{Z}^2$ . Each walk is a classical simple random walk, where  $e_1$  corresponds to stepping up and  $e_2$  corresponds to stepping down. Any two random walks starting at  $x \neq y \in \mathbb{Z}^2$  must coalesce eventually almost surely. This is because the difference  $X_x - X_y$  is a one-dimensional simple random walk that is almost surely recurrent to the origin. In contrast, the periodic system in Fig. 1 (also a measure-preserving ergodic  $\mathbb{Z}^2$  system) has bi-infinite trajectories.

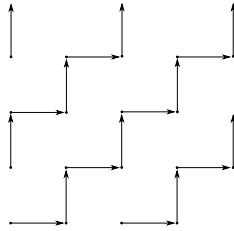


Figure 1: The space is  $\Omega = \{\omega_1, \omega_2\}$  with uniform measure. The arrows are given by  $\alpha(\omega_i) = e_i$  for  $i = 1, 2$ . The translation operators  $T^{e_i}$  simply swap between  $\omega_1$  and  $\omega_2$ .

In this paper, we completely classify the collective behavior of the trajectories in  $d = 2$  under mild assumptions. There is a behavioural *dichotomy* (Theorem 2.3): with probability 1,

1. the walks from all  $x, y \in \mathbb{Z}^2$  coalesce, and there exist no bi-infinite trajectories, or
2. a positive fraction of the walks in a configuration form bi-infinite trajectories that are themselves measure-preserving dynamical systems; all the walks have the same asymptotic direction (Theorem 2.8); no two bi-infinite trajectories coalesce (Theorem 2.5); and in systems with completely positive entropy (e.g. factors of iid systems), bi-infinite trajectories must “carry entropy” (Theorem 2.12).

All the nice properties that the bi-infinite trajectories possess are not shared by almost surely coalescing walks. For example, asymptotic direction is no longer guaranteed. We demonstrate this by constructing an explicit example (Theorem 2.13).

In higher dimensions ( $d > 2$ ), the bi-infinite trajectories—almost-sure coalescence dichotomy is not true. We construct an example (Corollary 2.15) in  $d = 3$  where almost surely,

- (i) every trajectory does not have an asymptotic direction,
- (ii) every configuration does not have bi-infinite trajectories, and
- (iii) we do not have almost sure coalescence.

## 1.1 Background

Our model is motivated by questions about the behavior of infinite geodesics in first- and last-passage percolation. Let  $\Omega = \{\omega_z \in \mathbb{R}\}_{z \in \mathbb{Z}^d}$  with product  $\sigma$ -algebra and a translation invariant measure  $\mathbb{P}$ . The  $\omega_z$  are called *weights* and they’re typically nonnegative random variables. Let  $X_{x,y}$  be a path from  $x$  to  $y$  and let the total weight of the path be the sum  $W(X_{x,y}) := \sum_{z \in X_{x,y}} \omega_z$ . Define the first-passage time from  $x$  to  $y$  to be

$$T(x, y) = \inf_{X_{x,y}} W(X_{x,y}).$$

The first-passage time  $T(x, y)$  satisfies a triangle inequality; if the weights are strictly positive, it defines a random metric on the lattice  $\mathbb{Z}^d$ . A geodesic for this random metric is a nearest-neighbor path that minimizes the passage time between every point that lies on it.

By considering the geodesic from 0 to  $ne_1$ , and then looking at a subsequence as  $n \rightarrow \infty$ , it’s clear that there is at least one semi-infinite geodesic from the origin. Furstenberg (communicated in Kesten [15] page 134) asked if there exist bi-infinite geodesics (bigeodesics) in first-passage percolation with iid weights. This question has not been answered completely. However, there are several partial answers under different assumptions on the so-called time-constant of first-passage percolation. We survey a few important results below.

Newman [20] was motivated by a connection between first-passage percolation and infinite ground states of the Ising model with nonnegative disorder in

$d = 2$ . In a seminal paper [18], Licea and Newman prove theorems about the non-existence of bigeodesics under strong assumptions on the so-called time-constant of first-passage percolation. For  $u \in S^1$ , define the time-constant  $g(u)$  as

$$g(u) = \lim_{n \rightarrow \infty} \frac{T(0, [nu])}{n}.$$

The limit exists almost surely and in  $L^1$  due to Kingman's subadditive ergodic theorem [16]. Even in the iid case, apart from general properties like convexity, very little is known about  $g(u)$ . The sub-level set  $\{u: g(u) \leq 1\}$  is known as the limit-shape. When the weights are stationary-ergodic (and even mixing with positive entropy), it's known that any bounded, convex, compact limit-shape is possible [13].

Licea and Newman assume that 1)  $g(u)$  is "uniformly curved", 2) the weights satisfy a property called finite-energy (see Burton and Keane [5]) and 3) the weights are iid, and have continuous distribution. Then, with probability 1, except for a deterministic Lebesgue measure 0 set of directions  $u \in S^1$ ,

1. there exist a geodesic from each point  $x \in \mathbb{Z}^2$  in direction  $u$ ,
2. geodesics from different points coalesce almost surely, and
3. there are no bigeodesics in direction  $(-u, u)$ .

In this and subsequent papers [18, 20], Newman realized the importance of so-called Busemann functions to describe the behavior of geodesics. A Busemann function is defined as the limit (if it exists)

$$B_u(x, y) = \lim_{n \rightarrow \infty} T(x, z_n) - T(y, z_n)$$

such that  $z_n/n \rightarrow u \in S^1$ . The first notable use of Busemann functions in first passage percolation was by Hoffman [14] to prove (concurrently with Garet and Marchand [10]) that there are at least two semi-infinite geodesics under no assumptions on the time-constant  $g(u)$ . Busemann functions have many useful properties, but from our perspective, the most useful property is that they *encode geodesic behavior in a stationary manner*.

Damron and Hanson built a theory of generalized Busemann functions in first-passage percolation [8]. Theirs and previous work are summarized in the survey by Auffinger et al. [3]. Associated with each direction  $u \in S^1$ , they construct a stationary function  $B_u(x, y)$  called a reconstructed Busemann function. By constructing the geodesics associated with these reconstructed Busemann functions, they obtain a directed geodesic graph  $G_u$  with vertices in  $\mathbb{Z}^2$  such that

1. each directed path is a geodesic,
2. If  $x \rightarrow y$  in  $G_u$  then  $B_u(x, y) = T(x, y)$ ,
3.  $G_u$  has no circuits even as an undirected graph,

4. each vertex has out degree 1.

The edges of this directed graph are the analogs of our arrow configurations  $\{\alpha(T^z\omega)\}$ . They then show that under the upward finite-energy assumption [2], the geodesic graph coalesces almost surely. Our dichotomy theorem classifies the behavior of such geodesic walks even when finite-energy does not hold (see Corollary 2.6). Similar results have recently been proved using different methods by Ahlberg and Hoffman [1]. In first-passage percolation on some hyperbolic graphs Benjamini and Tessera [4] have shown that bigeodesics exist.

Another motivation comes from the work of Georgiou et al. [11]. In first- and last-passage percolation, dual variational descriptions of the time-constant  $g(u)$  have recently been proved by Krishnan [17] and Georgiou et al. [12]. Here, the time constant is expressed as a minimization problem over *functions* instead of paths. It turns out that certain special minimizers of the formula are the Busemann functions we discussed earlier<sup>1</sup>.

In Theorem 2.3 of [11], they prove the last-passage version of the results of [8] in  $d = 2$ : If the last-passage time-constant  $g$  is differentiable at  $u \in S^1$ , there exists a stationary function  $B_u(\omega, x, y)$  that can be used to create a family of non-crossing geodesics going in direction  $u$  by following the arrows defined by

$$\alpha(x, \omega) = \operatorname{argmin}_{z=e_1, e_2} B_u(\omega, x, x+z). \quad (2)$$

Again, under the assumption of finite-energy, they prove almost sure coalescence of geodesics using the Licea-Newman argument. The property that one can create geodesics using (2) is rather special. In the general setting with stationary-ergodic weights, minimizers of the variational formulas with this property do not necessarily exist<sup>2</sup>. Nevertheless, one can still create paths from any minimizer using the recipe in (2). Our paper is an attempt to classify the behavior of these paths to better understand the behavior of the variational formulas of first- and last-passage percolation.

A different set of results that follow from theorems comes from a question asked by C. Hoffman: “On a semi-infinite geodesic, does the empirical measure of weights seen on the geodesic converge?” We interpret this as follows. Given a family of non-crossing geodesics  $\{X_x\}_{x \in \mathbb{Z}^d}$  and vertex weights  $\{\omega_z\}_{z \in \mathbb{Z}^d}$ , does the measure defined by  $n^{-1} \sum_{i=1}^n \delta_{\omega(X_x(i))}$  converge as  $n \rightarrow \infty$ ? Under an integrability condition, one can also ask if there is a limiting asymptotic weight/speed on a geodesic: does  $n^{-1} \sum_{i=1}^n \omega(X(i))$  converge? When bigeodesics exist (Theorem 2.8), these limits exist (the latter exists when  $\omega(0)$  is integrable on the bi-infinite geodesics). Using the example constructed in Theorem 2.13, we show that there is a geodesic walk where these limits do not exist.

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<sup>1</sup>From the point of view of stochastic homogenization, the Busemann functions are known as correctors.

<sup>2</sup>see Lions and Souganidis [19] in the context of continuum stochastic homogenization

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## 2 Main results

**Notation** We will generally use Greek or calligraphic letters to denote events  $\mathcal{E} \in \mathcal{F}$ . We write subsets of  $\mathbb{Z}^d$ , random or deterministic, using the Latin alphabet. We will frequently speak of a “configurations having a density of points with a certain property”. We explain what we mean by this in the following.

**Definition 2.1** (Rectangular subsets of  $\mathbb{Z}^d$ ). *Let the rectangle centered at  $x \in \mathbb{Z}^d$  with side lengths  $2N$  be*

$$\text{Rect}_x(N) = \prod_{i=1}^d [x_i - N, x_i + N].$$

The boundary of the any  $R \subset \mathbb{Z}^d$  is written as  $\partial R$  and consists of the set of points in  $R$  that have at least one point in  $R^c$  as a nearest neighbor.

A subset  $A \subset \mathbb{Z}^d$  has density if there is a number  $c \in [0, 1]$  such that  $N^{-d}|A \cap \text{Rect}_0(N)| \rightarrow c$ . Given an event  $\mathcal{E}$ , the ergodic theorem ensures that almost surely in every configuration the random subset  $A(\omega) = \{x \in \mathbb{Z}^d : T^x \omega \in \mathcal{E}\}$  occurs with density  $\mathbb{P}(\mathcal{E})$ . Motivated by this interpretation, we will sometimes abuse notation and write for a set  $A \subset \mathbb{Z}^d$  and an event  $\mathcal{E}$ ,

$$(A \cap \mathcal{E})(\omega) := \{x \in A : T^x \omega \in \mathcal{E}\}.$$

**Definition 2.2** (Coalescence of points). *Given a configuration, we say that the points  $x$  and  $y$  coalesce if the walks  $X_x$  and  $X_y$  coalesce in the future. We say we have almost sure coalescence if almost surely for all  $x, y \in \mathbb{Z}^d$ , the walks through  $x$  and  $y$  coalesce.*

**Theorem 2.3.** *In  $\mathbb{Z}^2$ , suppose*

1. *the walks have no loops, and*
2. *each walk intersects every line in the vertical direction at most a finite number of times.*

*Then, if there are no bi-infinite walks, we must have almost sure coalescence.*

**Remark 2.4.** Vertical lines may clearly be replaced with horizontal lines or diagonal lines ( $x = \pm y$ ).

We begin with the converse of Theorem 2.3.

**Theorem 2.5.** *In almost every configuration, bi-infinite walks do not coalesce.*

Theorem 2.5 is classical in first- and last-passage percolation: it says that non-crossing bigeodesics do not coalesce (see [18], Theorem 4.6 in [11] and Theorem 6.9 in [8]).

Next, we begin with a simple corollary of Theorem 2.3 in first-passage percolation. It completely classifies the behavior of all geodesic families that cross-lines in the sense of Theorem 2.3. Damron and Hanson constructed such families in [8]. They have two sets of assumptions:

**A1.**

1.  $\Omega = \{\omega_e\}_{e \in E}$  is the set of non-negative edge-weights on  $E$ , the nearest-neighbor edges on  $\mathbb{Z}^2$ .  $\mathbb{P}$  is the product measure.
2. The weights satisfy the Cox-Durrett condition [7]:

$$\mathbb{E} \left[ \left( \min_{e \in \{\pm e_1, \pm e_2\}} \omega_e \right)^2 \right] < \infty.$$

3.  $\mathbb{P}(\omega_e = 0) < p_c = \frac{1}{2}$ , the critical probability for bond-percolation on  $\mathbb{Z}^2$  (see Theorem 1.15 in Kesten [15]).
4. The weights have continuous distribution; i.e., they take any fixed value with zero probability.

**A2.**

1.  $(\Omega, \mathcal{F}, \mathbb{P})$  is an ergodic  $\mathbb{Z}^2$  system of weights.
2.  $\mathbb{P}$  has all the symmetries of  $\mathbb{Z}^2$ .
3.  $\mathbb{E}[\omega_e^{2+\epsilon}] < \infty$  for some  $\epsilon > 0$ .
4. the limit shape for  $\mathbb{P}$  is bounded.
5.  $\mathbb{P}$  has unique passage times; i.e., if  $X_1$  and  $X_2$  are two finite paths,  $W(X_1) \neq W(X_2)$  almost surely.

The boundedness of the limit-shape is guaranteed if, for example,  $\omega_e \geq a > 0$  almost surely. Under assumption **A1** or **A2** along with the finite-energy assumption, they prove that the non-crossing geodesic family they construct must coalesce almost surely [8, Theorem 1.10]. The next corollary completes the picture when finite-energy does not hold.

**Corollary 2.6.** Let  $\mathbb{G}$  be the directed, non-crossing geodesic graph constructed in [8, Prop. 5.1 and Prop 5.2] under assumption **A2**. Then, infinite paths on  $\mathbb{G}$  either coalesce almost surely or form bi-infinite trajectories.

Proposition 5.1 of [8] shows that there exists an infinite path from each  $x \in \mathbb{Z}^2$ . Proposition 5.2 shows that these paths must be non-crossing. Together, Prop. 5.2, Theorem 5.3 and Lemma 6.2 of [8] verify assumptions 1 and 2 of Theorem 2.3; in particular, it shows that the family of geodesics is asymptotically directed in a sector of angular size at most  $\pi/2$ . Hence, Corollary 2.6 follows.

**Remark 2.7** (Last-passage percolation). The geodesic families in last-passage percolation constructed in [12] uses a result from queuing theory that assumes iid weights. Since iid weights satisfy the upward finite-energy property [8], they coalesce almost surely. If one were to construct stationary, non-crossing families of geodesics in last-passage percolation without assuming finite energy, our dichotomy theorem would imply something non-trivial.

The arrows induce a map  $T_\alpha$  along walks defined by

$$T_\alpha \omega = T^{\alpha(\omega)} \omega.$$

The  $T_\alpha$  map is neither measure preserving nor invertible in general. Along bi-infinite trajectories, however, it's both invertible and measure preserving. This observation and Theorem 2.5 are used in the next theorem. Let  $\mathcal{S}$  be the event that the origin is in a bi-infinite trajectory. For any  $A \in \mathcal{F}$ , let  $\mathbb{P}_\alpha(A) = \mathbb{P}(A \cap \mathcal{S})$  and  $\Omega_\alpha = \Omega \cap \mathcal{S}$  to obtain the measure space  $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$ .

**Theorem 2.8.** *The bi-infinite trajectories form a measure-preserving  $\mathbb{Z}$ -system  $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha, T_\alpha)$ . Hence, all bi-infinite trajectories have asymptotic direction. When  $d = 2$ , all walks in a configuration have the same asymptotic direction. When the  $\mathbb{Z}^2$  system  $(\Omega, \mathcal{F}, \mathbb{P}, T)$  is ergodic, this direction is deterministic.*

**Remark 2.9.** Suppose we have weights  $(\{\omega_x\}_{x \in \mathbb{Z}^d}, \mathcal{F}, \mathbb{P})$  and a stationary set of geodesic walks  $\{X\}_{z \in \mathbb{Z}^d}$  (formed using the Damron-Hason procedure, say [8]). Then, if  $X_0$  is on a bigeodesic, for any set  $A \in \mathbb{R}$ ,  $n^{-1} \sum_{i=1}^n \delta_{\omega_{X_0(i)}}(A) \rightarrow \mathbb{P}_\alpha(\omega \in A)$  converges almost surely and in  $L^1$ .

Bi-infinite trajectories are rather special. When there is sufficient randomness in the system, it's not possible for all trajectories to be bi-infinite. For the next two results, we restrict to up-right paths in  $d = 2$ .

**Proposition 2.10.** *In  $d = 2$ , suppose we have non-trivial up-right walks  $(\mathbf{A} = \{e_1, e_2\})$ , and that  $T^{e_2 - e_1}$  is ergodic. Then, all trajectories cannot be bi-infinite.*

In the periodic system in Fig. 1, translation in the anti-diagonal direction is certainly not ergodic. This allows every trajectory to be bi-infinite.

Proposition 2.10 can also be proved if the arrow configurations have positive entropy-rate. For any measure  $\nu$  supported on a finite alphabet  $A = \{a_1, \dots, a_n\}$ , the Shannon entropy is defined as  $H(\nu) = -\sum_{i=1}^n \nu(a_i) \log \nu(a_i)$ . Let  $\Omega := \mathbf{A}^{\mathbb{Z}^d}$  be the space of arrow configurations, and let  $\mathcal{B}$  be the Borel  $\sigma$ -algebra. Given a subset  $M \subset \mathbb{Z}^d$  and a cylinder  $\mathcal{E} \subset \mathbf{A}^M$ , the probability



measure is naturally given by  $\mathbb{P}(\{\alpha(T^z)\}_{z \in M} \in \mathcal{E})$ . Let  $\pi_M: \mathbf{A}^{\mathbb{Z}^d} \rightarrow \mathbf{A}^M$  be the coordinate projection map.

The entropy-rate of  $(\mathbf{A}^{\mathbb{Z}^d}, \mathcal{B}, \mathbb{P}, T)$  is defined as usual by

$$h(\mathbf{A}^{\mathbb{Z}^d}, \mathbb{P}) = \inf_{M \subset \mathbb{Z}^d} |M|^{-1} H(\mathbb{P} \circ \pi_M^{-1}).$$

**Proposition 2.11.** *In  $d = 2$ , suppose we have up-right walks  $(\mathbf{A} = \{e_1, e_2\})$ , and assume that the entropy-rate  $h(\Omega, \mathcal{F}, \mathbb{P}, \{T\}_{z \in \mathbb{Z}^2})$  is positive. Then all trajectories cannot be bi-infinite.*

In the context of first- and last-passage percolation, the weights  $\{\omega_x\}_{x \in \mathbb{Z}^d}$  are often assumed to have the finite-energy property [18, 20, 21]. This is a very useful, but rather strong assumption. Further, factors of finite-energy systems don't inherit finite-energy.

A more natural assumption from the point of view of ergodic theory is *completely positive entropy*. A system has completely positive entropy if all of its factors have positive entropy. In particular, when  $(\Omega, \mathcal{F}, \mathbb{P})$  is a product space with product measure, then it and all its factors have completely positive entropy [22]. Moreover, the finite-energy assumption *implies* completely positive entropy. The proof is in Section 6.

Under the assumption of completely positive entropy, we cannot recover the results of [18, 20, 21], where the finite-energy assumption is used to show that non-crossing semi-infinite trajectories coalesce almost surely. This implies that bi-infinite trajectories do not exist. Nevertheless, under the completely positive entropy assumption, the bi-infinite trajectories must carry entropy.

**Theorem 2.12.** *In  $d \geq 2$ , let the paths be directed  $(\mathbf{A} = \{e_1, \dots, e_d\})$  and let  $(\mathbf{A}^{\mathbb{Z}^d}, \mathcal{B}, \mathbb{P}, T)$  have completely positive entropy. Let  $(\mathbf{A}^{\mathbb{Z}}, \mathcal{B}, \mathbb{P}_\alpha, T_\alpha)$  be the  $\mathbb{Z}$ -system of arrows configurations on bi-infinite trajectories (see Theorem 2.8). Then, the entropy-rate of  $(\Omega_\alpha, \mathcal{B}, \mathbb{P}_\alpha, T_\alpha)$  on each ergodic component of  $\mathbb{P}_\alpha$  is positive almost surely; i.e.,*

$$h(\Omega_\alpha, \theta_\omega) > 0 \quad \text{a.s } \omega,$$

where  $\theta_\omega$  is an ergodic component of  $\mathbb{P}_\alpha$  (see (4) for the ergodic decomposition).

After establishing all these nice properties of the bi-infinite trajectories, we show that in the case of almost sure coalescence, none of these properties need to hold. We do this by constructing an example in  $\mathbb{Z}^2$  using a classical cutting and stacking construction, which were initiated by Chacon [6].

**Theorem 2.13.** *There exists an ergodic  $\mathbb{Z}^2$  dynamical system,  $(\Omega, \mathcal{F}, \mathbb{P})$ , where we have almost sure coalescence but almost every trajectory does not have an asymptotic direction.*

**Remark 2.14.** This example can be shown to create a first- or last-passage percolation model with geodesic walks that coalesce almost surely, but have no

asymptotic direction. This shows that the empirical measure of weights seen on geodesics need not converge in general.

We then use Theorem 2.13 to construct examples in  $\mathbb{Z}^3$  where we have neither almost sure coalescence nor bi-infinite trajectories. In other words, the dichotomy theorem in  $\mathbb{Z}^2$  no longer holds.

**Corollary 2.15** (of Theorem 2.13). *There exists an ergodic  $\mathbb{Z}^3$  system, where almost surely,*

- (i) every trajectory does not have an asymptotic direction,
- (ii) every configuration does not have bi-infinite trajectories, and
- (iii) we do not have almost sure coalescence.

### 3 Non coalescence implies bi-infinite trajectories

Before proceeding with the proof of Theorem 2.3, we prove an elementary lemma that we'll use repeatedly.

**Lemma 3.1.** *Let  $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Z}^d)$  be an ergodic  $\mathbb{Z}^d$  system, and let  $M(\omega) \subset \mathbb{Z}^d$  be a random set of points such that  $|M(\omega)| \geq 1$  occurs with positive probability. Then, almost surely,  $M(\omega)$  must have density  $\rho \in (0, 1) > 0$ .*

*Proof.* There exists  $N > 0$  so that  $\mathbb{P}(|M(\omega) \cap B_0(N)| \geq 1) > c > 0$ . By the ergodicity of  $(\Omega, \mathbb{P}, \mathbb{Z}^d)$  for almost every  $\omega$  the density is at least  $\frac{c}{(2N+1)^2}$ .  $\square$

In a given configuration  $\omega$ , we say that a point  $x \in \mathbb{Z}^d$  has a *past of length*  $n$  if there is a  $z$  such that  $X_z(n, \omega) = x$ . Define the random set of points

$$P_n(\omega) := \{x \in \mathbb{Z}^2 : x \text{ has a past of length } n\}.$$

The set of points in bi-infinite trajectories are  $\cap_{n \geq 0} P_n(\omega)$ . Theorem 2.3 follows directly from the following lemma, which shows that the probability that the origin is in a trajectory of length  $n$  is bounded uniformly below by a constant  $\rho > 0$ .

**Lemma 3.2.** *It suffices to show that under assumptions 1 and 2 of Theorem 2.3, there exists a constant  $\rho > 0$  so that for all  $n$  the set of points with a past of length  $n$  has density at least  $\rho$ .*

Since the walk through the origin must intersect the vertical line through the origin at a last time, the “last-crossing” points must have positive density. Let  $\text{Vert}_a^k = \{a\} \times [-k, k]$  and write  $\text{Vert}_a$  for the entire vertical line with  $e_1$  coordinate  $a$ . Let  $L^+(\omega)$  be the set  $(a, b) \in \mathbb{Z}^2$  that lie on a walk, say,  $X(\omega)$ , so that a connected component of  $X(\omega) \setminus (a, b)$  has all of its  $x$ -coordinates strictly greater than  $a$  and crosses all vertical lines  $\text{Vert}_c$  where  $c > a$ . Let  $L^-(\omega)$  be the analogous set where the walk crosses all vertical lines  $c < a$ .

By assumption 1 in Theorem 2.3, there must be at least one point in either  $L^+(\omega) \cup L^-(\omega)$ . For convenience, we assume that  $L(\omega) = L^+(\omega)$  must have density for the remainder of the argument. The  $L^-$  case can be included with a straightforward modification.

**Lemma 3.3.**  *$L(\omega)$  has density  $\xi > 0$  almost surely.*

This follows directly from Lemma 3.1.

The next lemma says that if one walk is below the other when they're at their last crossing points on a line, they must continue to maintain this ordering in the future. Consider two points  $x, y \in L(\omega) \cap \text{Vert}_a$ . The lemma follows from the fact that lines separate the plane and that the geodesics from  $x$  and  $y$  will never cross  $\text{Vert}_a$  again. We say that  $x < y \in \text{Vert}_a$  if  $x \cdot e_2 < y \cdot e_2$ .

**Lemma 3.4.** *Let  $x, y \in \mathbb{Z}^2$  and  $p \in \mathbb{Z}$  such that  $x < y \in \text{Vert}_p \cap L(\omega)$ , and that  $X_x$  and  $X_y$  do not coalesce. Then if  $p < q \in \mathbb{Z}$ , and  $(q, i) \in X_x$  and  $(q, j) \in X_y$  we must have  $i > j$ .*

Consider the set of points in  $L(\omega) \cap \text{Vert}_a$  that don't coalesce with at least one other point in  $L(\omega) \cap \text{Vert}_a$ . Vertically order these points as  $\dots < x_{-1} < x_0 < x_1 < \dots$ , such that  $x_0$  is the point with the smallest  $e_2$  coordinate in absolute value (with some tie-breaking rule). Let  $S_a \subset L(\omega) \cap \text{Vert}_a$  be the set of even last-crossing points  $\{x_{2i}\}_{i \in \mathbb{Z}^2}$ . We call  $S_a(\omega)$  a *separating set*.

**Corollary 3.5.** The walks from two distinct points  $x, y \in S_a$  do not coalesce to the right of  $\text{Vert}_a$ .

**Lemma 3.6.** *There exists  $\zeta_1, \zeta_2 > 0$  so that exists a set of  $a \in \mathbb{Z}$  with density at least  $\zeta_2$  so that for each such  $a$  there exists a separating set  $S_a$  with density at least  $\zeta_1$  on the vertical line through  $a$ .*

*Proof.* Almost surely, there must be at least a pair of points  $x, y \in L(\omega)$  that don't coalesce. By Lemma 3.1, there must be a density of such points. Then, in a rectangle  $\text{Rect}_0(N, N)$ , there must be at least  $cN^2$  such points with high probability for some  $c > 0$ .

Now by the ergodic decomposition of the  $\mathbb{Z}$  action  $\omega \rightarrow T^{(0,1)}\omega$ , for almost every  $\omega$  there is a density of  $\{j \in \text{Vert}_0(\omega) \cap L(\omega)\}$ . So for almost every  $\omega$ , for every  $a$  we have that  $\{j \in \text{Vert}_a(\omega) \cap L(\omega)\}$  has a density. By the previous paragraph, a positive density set of  $a$  need to have that this density is non-zero. On these positive density lines, the points in  $S_a$  must have at least half the density, by construction. Therefore, the statement of the lemma follows.  $\square$

**Lemma 3.7.** *Let  $S_k(\omega)$  be a separating set. For all  $m, n > 0$  we have*

$$|P_n(\omega) \cap \partial \text{Rect}_{k,0}(n, m+n)| \geq |S_k(\omega) \cap \text{Vert}_a(m)|.$$

*Proof.* By our assumption if  $(k, c) \in S_k$  and  $c \in [-m, m]$ , then the walk  $X_{(k,c)}$  must cross  $\partial \text{Rect}_{(k,0)}(n, m+n)$ . Clearly,  $X_{(k,c)}$  must take at least  $n$  steps before it crosses  $\partial \text{Rect}_{(k,0)}(n, m+n)$ , and hence the crossing point must be in  $P_n$ . By Corollary 3.5 any distinct  $c$  cross at distinct points in  $\partial \text{Rect}_{(k,0)}(n, m+n)$ .  $\square$

**Corollary 3.8.** Let  $\zeta_1$  be as in Lemma 3.6. For all  $\epsilon > 0$  there exists  $N_0$  so that for all  $N > N_0$ ,

$$\mathbb{P} \left( \left\{ \omega : |P_n(\omega) \cap \partial \text{Rect}_0(N, N)| > \frac{\zeta_1}{8} N \right\} \right) > (1 - \epsilon).$$

*Proof.* By the previous lemma it suffices to show that for each  $\epsilon > 0$  and  $n$  there exist  $N > n$  and  $|k| \leq N - n$  so that  $|S_k(\omega) \cap \text{Vert}_k(N - n)| > \frac{\zeta_1}{8} N$  with probability at least  $1 - \epsilon$ . This follows from Lemma 3.6 and the ergodicity of our  $\mathbb{Z}^2$  system.  $\square$

By Lemma 3.2 the next lemma establishes Theorem 2.3.

**Lemma 3.9.** Let  $\zeta_1$  be as in Lemma 3.6. There exists  $M_0$  so that

$$\mathbb{P} \left( \left\{ \omega : |P_n \cap \text{Rect}_0(M, M)| > \frac{\zeta_1}{32} M^2 \right\} \right) > \frac{1}{4}$$

for all  $M > M_0$ .

*Proof.* Let  $\mathcal{G}_N := \{\omega : |P_n \cap \partial \text{Rect}_0(N, N)| > \frac{\zeta_1}{8} N\}$  be the event that there is a rectangle of side-length  $2N + 1$  centered at the origin whose boundary has a significant number of points in  $P_n(\omega)$ . By the previous corollary, there exists  $N$  so that  $\mathbb{P}(\mathcal{G}_N) > \frac{1}{2}$ . Therefore, for  $M > N$ , with probability at least  $\frac{1}{4}$ , we must have

$$|\text{Rect}_0(M, M) \cap \mathcal{G}_N| \geq \frac{1}{4}(2M + 1)^2.$$

Hence, for each point  $(p, q) \in \text{Rect}_0(M - N, M - N) \cap \mathcal{G}_N$  we have at least  $\frac{\zeta_1}{8} N$  points in the boundary of the rectangle around  $(p, q)$  that are in  $P_n$ . Each such point in  $P_n$  occurs with multiplicity at most  $|\partial \text{Rect}_0(N, N)| = 8N$  for different points in  $\text{Rect}_0(M - N, M - N) \cap \mathcal{G}_N$ . Thus, we obtain the following lower bound on points with past length  $n$  inside a rectangle of size  $M$ . With probability at least  $1/4$ ,

$$\begin{aligned} & |P_n \cap \text{Rect}_0(M, M)| \\ & \geq \frac{(|\mathcal{G}_N \cap \text{Rect}_0(M, M)| - |\text{Rect}_0(M, M) \setminus \text{Rect}_0(M - N, M - N)|) \frac{\zeta_1}{8} N}{4N} \\ & \geq \frac{((2M + 1)^2 \frac{1}{4} - 4MN) \zeta_1}{32N^2}. \end{aligned}$$

By choosing  $M$  sufficiently large (given  $N$ ) the lemma follows.  $\square$

## 4 Bi-infinite trajectories

In this section we assume that we do not have almost sure coalescence and therefore have bi-infinite trajectories. All results in this section except Theorem 2.8 hold for  $\mathbb{Z}^d$  actions with any  $d$ .

**Definition 4.1.** We say that a point  $x \in \mathbb{Z}^d$  is cataclysmic if it is a point of coalescence of two distinct bi-infinite trajectories in a configuration.

**Lemma 4.2.** Given any bounded rectangle,  $R$ , the number of points in  $\partial R$  crossed by a bi-infinite trajectory is at least the number of cataclysmic points in  $R$ .

*Proof.* Consider two bi-infinite trajectories to be equivalent if they coalesce together at cataclysmic points in  $R$ . Each equivalence class of trajectories forms a tree. It suffices to prove the result on each equivalence class of bi-infinite trajectories. Fix an equivalence class and put an order on it. After two bi-infinite trajectories coalesce, consider the coalesced trajectory to be the largest trajectory that has coalesced. At each cataclysmic point visited by the equivalence class, select the smallest of the trajectories that coalesce with it. By construction, each trajectory in the equivalence class is selected at most once. Repeating this selection procedure for all equivalence classes, we obtain at least the number of the cataclysmic points many bi-infinite trajectories. Each selected trajectory gives at least one distinct crossing of the boundary because the region is bounded and the trajectories have an infinite past.  $\square$

*Proof of Theorem 2.5.* Assume that at least one cataclysmic point occurs with positive probability. By Lemma 3.1, cataclysmic points occur with density  $\rho > 0$ . Choose  $N$  so that  $\rho N^d > 2|\partial[0, N]^d|$ . Then with positive probability, there are at least  $\frac{\rho}{2}N^d$  cataclysmic points in  $[0, N]^d$ . By the previous lemma, each cataclysmic point can be associated with a distinct point on the boundary of  $\partial[0, N]^d$ . However, there aren't enough points on  $\partial[0, N]^d$  to accomodate them all and this is a contradiction.  $\square$

**Remark 4.3.** Using the ergodic decomposition we may relax our assumption to  $\mathbb{Z}^d$ -preserved measures.

## 4.1 Asymptotic direction

Let  $\mathbb{P}_\alpha$  be a measure on  $\mathcal{S}$  defined by  $\mathbb{P}_\alpha(\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{S})$  for  $\mathcal{A} \in \mathcal{F}$ .

**Proposition 4.4.**  $T_\alpha$  is a  $\mathbb{P}_\alpha$  measure preserving  $\mathbb{Z}$  action.

*Proof.* Since from Theorem 2.5 the bi-infinite trajectories cannot coalesce,  $T_\alpha$  must be almost surely invertible on the bi-infinite points  $\mathcal{S}$ . For each  $y \in \mathbf{A}$ , let

$$\mathcal{V}_y = \{\omega \in \mathcal{S} : \alpha(T_\alpha^{-1}\omega) = y\}.$$

Clearly,  $\{\mathcal{V}_y\}_{y \in \mathbf{A}}$  is a partition of  $\mathcal{S}$ . Thus, for any  $\mathcal{E} \subset \mathcal{S}$ ,

$$\begin{aligned} \mathbb{P}_\alpha(T_\alpha^{-1}(\mathcal{E})) &= \mathbb{P}(T_\alpha^{-1}(\mathcal{E}) \cap \mathcal{S}) = \mathbb{P}(T_\alpha^{-1}(\mathcal{E} \cap \mathcal{S})) \\ &= \sum_{y \in \mathbf{A}} \mathbb{P}(T_\alpha^{-1}(\mathcal{E} \cap \mathcal{V}_y)) = \sum_{y \in \mathbf{A}} \mathbb{P}(T^{-y}(\mathcal{E} \cap \mathcal{V}_y)) \\ &= \sum_{y \in \mathbf{A}} \mathbb{P}_\alpha(\mathcal{E} \cap \mathcal{V}_y) = \mathbb{P}_\alpha(\mathcal{E}) \end{aligned}$$

The first equality is by the definition of  $\mathbb{P}_\alpha$ . The second is because  $\mathcal{S}$  is  $T_\alpha$  invariant. The third follows from the definition of the  $\mathcal{V}_y$ . The fourth is because on  $\mathcal{V}_y$  the  $T_\alpha^{-1} = T^{-y}$ , and the fifth is because each  $T^{-y}$  is  $\mathbb{P}$  measure preserving.  $\square$

**Lemma 4.5.** *There is an ergodic decomposition of  $(\mathcal{S}, \mathbb{P}_\alpha, T_\alpha)$ . This means almost every bi-infinite trajectory is ergodic (for some measure).*

Let  $F = \{a_1, \dots, a_m\}$  be any finite set of symbols in  $\mathbf{A}$ . Then, we say  $F$  appears on a bi-infinite path if  $\{\alpha(T_\alpha^{n+i}(\omega))\}_{i=1}^m = F$  for some  $n$ .

**Corollary 4.6.** There is a block-dependent frequency for each finite block on every bi-infinite path.

This follows from the previous lemma and the Birkhoff ergodic theorem.

*Proof of Theorem 2.8.* Proposition 4.4 shows that  $(\mathcal{S}, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$  is a measure-preserving dynamical system. Applying the previous corollary to each singleton  $F_i = \{e_i\}$ ,  $i = 1, \dots, d$  shows that all bi-infinite trajectories have asymptotic direction.

In dimension 2, let  $\theta_\omega$  be an ergodic component of  $\mathbb{P}$  such that  $\theta_\omega(\mathcal{S}) > 0$ . Assume for the sake of contradiction that there is no common asymptotic direction for configurations drawn from  $\theta_\omega$ . Then, there exists  $c$  so that a  $\mathbb{P}_\alpha$ -positive measure set of points are on bi-infinite trajectories with slope at least  $c$  and a  $\mathbb{P}_\alpha$ -positive measure set of points are on bi-infinite trajectories with slope strictly less than  $c$ . Call these sets  $\mathcal{S}_1$  and  $\mathcal{S}_2$  respectively. By the ergodicity of  $\theta_\omega$ , almost surely, there exist  $i, j$  and  $k, \ell$  with  $i < k$  and  $j > \ell$  so that  $T^{(i,j)}\omega \in \mathcal{S}_2$  and  $T^{(k,\ell)}\omega \in \mathcal{S}_1$ . The bi-infinite trajectory through  $i, j$  must coalesce with the bi-infinite trajectory through  $k, \ell$ . By Theorem 2.5, this must have zero probability. In case  $\mathbb{P}$  is ergodic, this direction is deterministic.  $\square$

Next, we prove Prop. 2.10, which shows that with a little “extra” randomness bi-infinite trajectories cannot occupy all points on the lattice. The next proposition is proved under the assumption that the walks are directed, i.e., the arrow directions are  $\mathbf{A} = \{e_1, e_2\}$ .

*Proof of Prop. 2.10.* Let  $\mathcal{E}_i$  be the event that  $\alpha(\omega) = e_i$  for  $i = 1, 2$ . The walks from  $e_1 - e_2$  and 0 cannot coalesce since they’re both in bi-infinite trajectories; this would contradict Theorem 2.5. Therefore  $T^{e_2 - e_1}\mathcal{E}_2 \cap \mathcal{E}_1 = \phi$ . By the ergodicity of  $T^{e_2 - e_1}$ , this means that  $\mathbb{P}(\mathcal{E}_2) = 1$ , contradicting the assumption that  $\alpha$  is non-trivial.  $\square$

## 4.2 Positive entropy

Proposition 2.10 shows that  $\mathbb{P}(\mathcal{S}) < 1$  in any system where  $T^{e_2 - e_1}$  is ergodic. We prove the same result in Prop. 2.11 for systems with positive entropy as a segue into our main theorem about systems with completely positive entropy.

In this section, let  $\Omega = \mathbf{A}^{\mathbb{Z}^d}$  be the space of arrow configurations and let  $\mathcal{B}$  be the product Borel  $\sigma$ -algebra on it. If a finite-alphabet  $\mathbb{Z}^d$  system has positive entropy, the Shannon-MacMillan theorem applies. We first state a corollary of the general Shannon-MacMillan theorem for non-ergodic measures that we state later (Theorem 4.8).

**Corollary 4.7** (Shannon-McMillan for ergodic measures). Let  $(\mathbf{A}^{\mathbb{Z}^d}, \mathcal{B}, \mathbb{P}, T)$  be a measure-preserving ergodic  $\mathbb{Z}^d$  system with entropy-rate  $h$ . For any  $\epsilon > 0$ , there is a large  $L$  such that whenever  $R \subset \mathbb{Z}^2$  is a rectangle of minimal side-length  $L$ , there is a set  $\mathcal{Y} \subset \mathbf{A}^R$  such that  $\mathbb{P}(\mathcal{Y}) > 1 - \epsilon$ , and for every  $a \in \mathcal{Y}$ ,

$$e^{-(h+\epsilon)L^d} < \mathbb{P}(a) < e^{-(h-\epsilon)L^d},$$

where  $\mathbb{P}(a) = \mathbb{P}(\pi_R^{-1}(a))$  with  $\pi_R: \mathbf{A}^{\mathbb{Z}^d} \rightarrow \mathbf{A}^R$  being the coordinate projection map.

This implies, in particular, that  $\mathcal{Y}$  must have exponentially many elements since

$$|\mathcal{Y}|e^{-(h-\epsilon)L^d} \geq \sum_{a \in \mathcal{Y}} \mathbb{P}(a) \geq (1 - \epsilon). \quad (3)$$

*Proof of Prop. 2.11.* Suppose for the sake of contradiction that the set of bi-infinite points has full measure. Then almost surely, there is a bi-infinite trajectory through every point on  $\mathbb{Z}^2$ .

Consider  $R \subset \mathbb{Z}^2$ , a rectangle of side length  $L$  that is aligned with the vector  $u = e_1 + e_2$ . More precisely, let  $R = \cup_{i=0}^{L-1} R_i$  be the union of lines  $R_i = \cup_{k=-L/2}^{L/2} \{iu + kv\}$ , where  $v = -e_1 + e_2$ .

We will count the number of configurations in  $\pi_R(\mathbf{A}^{\mathbb{Z}^d})$  and use Corollary 4.7 to produce a contradiction. Let  $x \in R_0$ , a point on the boundary of  $R$ . There are two possibilities for the next step of the walk  $X_x$ ,  $x + e_1$  or  $x + e_2$ . Suppose first that  $\alpha(x) = e_1$ . The point  $x + e_2$  must have an ancestor in  $R_0$ , for if not, there is no bi-infinite trajectory passing through it and this contradicts our assumption. Therefore,  $x + v \in R_0$  must be the ancestor of  $x + e_2$ ; i.e.,  $\alpha(x + v) = e_1$ . Similarly, we must have  $\alpha(x - v) = e_1$ . Otherwise, the bi-infinite trajectories from  $x$  and  $x - v$  would coalesce and contradict Theorem 2.5.

Thus, fixing  $\alpha$  on any single point  $x \in R_0$  determines  $\alpha$  on  $x - v$  and  $x + v$ , and thus, on all of  $R_0$ . So a single bi-infinite trajectory of length  $L$  starting from  $x \in R_0$  determines  $\alpha$  at one point on each  $R_i$ , and hence by the previous argument, on all points in  $R_i$ ,  $i = 1, \dots, d$ . There are most  $2^L$  different bi-infinite trajectories starting from a single point in the alphabet  $\mathbf{A}$ , and therefore, the total number of allowed configurations in  $R$  must satisfy  $|\pi_R(\mathbf{A}^{\mathbb{Z}^d})| \leq 2^L$  almost surely. Since  $h(\mathbf{A}^{\mathbb{Z}^d}, \mathbb{P}) > 0$ , this contradicts (3), which says that  $|\pi_R(\Omega)| \geq \frac{1}{2}e^{h(\Omega, \mathbb{P})L^2}$  with high probability for all large enough  $L$ .  $\square$

Next, we prove Theorem 2.12, which states that the bi-infinite trajectories must carry some of the entropy of the system when the system has completely

positive entropy. Completely positive entropy means that all nontrivial factors of the system have positive entropy. Generally, geodesic dynamics is studied under the assumption of finite-energy, which is a rather strong condition [18] and implies completely positive entropy (see Section 6). Let

$$\Omega_\alpha = \{(\alpha(T_\alpha^i \omega))_{i=-\infty}^\infty : \omega \in \mathcal{S}\}$$

be the set of all arrow configurations generated by the bi-infinite trajectories. There is a natural projection operator  $\pi_\alpha : \mathcal{S} \rightarrow \Omega_\alpha$ . For any  $a < b \in \mathbb{Z}^+$ , let

$$\pi_\alpha^{(a,b)}(\omega) = (\alpha(T_\alpha^i \omega))_{i=a}^b.$$

For  $(a, b) = (-\infty, \infty)$ , we will drop the superscript and write  $\pi_\alpha$ . For any subset  $\mathcal{A} \subset \Omega_\alpha$ , let  $\mathbb{P}_\alpha(\mathcal{A}) = \mathbb{P}(\pi_\alpha^{-1}(\mathcal{A}))$ .  $(\Omega_\alpha, \mathcal{B}, \mathbb{P}_\alpha, T_\alpha)$  is a measure-preserving dynamical system; indeed, it's a factor of the system defined in Theorem 2.8. Since this system may or may not be ergodic, Theorem 2.12 needs the generalized version of the Shannon-Macmillan theorem.

Let  $\mathcal{I}$  be the invariant  $\sigma$ -algebra of  $T_\alpha$ . Since  $(\Omega_\alpha, \mathcal{B}, \mathbb{P}_\alpha)$  can be turned into a separable metric space, it has a regular conditional probability and the following ergodic decomposition:

$$\mathbb{P}_\alpha = \int \theta_\omega \mathbb{P}_\alpha(d\omega),$$

where  $\theta_\omega$  is the conditional probability given  $\mathcal{I}$ .  $\theta_\omega$  is an invariant, ergodic measure for  $T_\alpha$  almost surely. The entropy-rate of this system is defined as

$$h(\Omega_\alpha, \mathbb{P}_\alpha) = \int h(\Omega_\alpha, \theta_\omega) \mathbb{P}_\alpha(d\omega). \quad (4)$$

The generalized Shannon-MacMillan theorem relates this entropy rate to the information function. This is a straightforward consequence of the usual Shannon-Macmillan Theorem and the ergodic decomposition.

**Theorem 4.8.** *[generalized Shannon-MacMillan] Let  $(\mathbf{A}^{\mathbb{Z}^d}, \mathcal{B}, \mathbb{P}, T)$  be a measure preserving  $\mathbb{Z}^d$  system and  $R \subset \mathbb{Z}^d$  be a finite rectangle. Let the information function  $f_R : \mathbf{A}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$  be*

$$f_R(\omega) = \begin{cases} -\log \mathbb{P}(\pi_R^{-1}(\pi_R(\omega))) & \mathbb{P}(\pi_R^{-1}(\pi_R(\omega))) > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then,

$$\lim_{|R| \rightarrow \infty} \|\pi_R^{-1} f_R(\omega) - h(\Omega_\alpha, \theta_\omega)\|_1 = 0,$$

where  $\theta_\omega$  is an ergodic component of  $\mathbb{P}$ , and the notation  $|R| \rightarrow \infty$ , means that the length of the smallest side of the  $R$  goes to infinity.



It follows from the Chebyshev inequality that for any  $\epsilon > 0$ , there is an  $L$  large enough such that for all rectangles  $R$  with minimal side-length larger than  $L$ , there exists a set  $\mathcal{Y} \subset \mathbf{A}^{\mathbb{Z}^d}$  with  $\mathbb{P}(\mathcal{Y}) \geq 1 - \epsilon$  and for all  $\omega \in \mathcal{Y}$ ,

$$e^{-(h(\theta_\omega) + \epsilon)|R|} \leq \mathbb{P}(\pi_R^{-1}(\pi_R(\omega))) \leq e^{-(h(\theta_\omega) - \epsilon)|R|}. \quad (5)$$

If  $\mathbb{P}$  is ergodic, Corollary 4.7 follows from (5).

Theorem 2.12 states that  $h(\Omega_\alpha, \theta_\omega) > 0$   $\mathbb{P}_\alpha$ -almost surely. The idea behind the proof is to assume for the sake of contradiction that there exist a subset  $\mathcal{C}$  of bi-infinite trajectories that have zero entropy. We then “recode” the arrow configurations so that the zero-entropy bi-infinite trajectories completely determine the arrow configurations. This new recoded system is a factor of the original system, and due to the completely positive entropy assumption, the recoded arrow configurations must also have positive entropy. Since all the entropy in the system is concentrated on the bi-infinite trajectories in the set  $\mathcal{C}$ , they could not have had zero entropy to start with.

The recoding lemma needs a few definitions. Fix any non-trivial  $\mathcal{C} \subset \Omega$ , and let  $C(\omega)$  be the set of points  $z \in \mathbb{Z}^d$  with smallest  $L^1$  distance from  $\mathbf{0}$  such that  $\omega(z) \in \mathcal{C}$ . To break ties, for any  $C \subset \mathbb{Z}^d$ , let  $\ell(C)$  pick out the smallest element in  $C$  ordered lexicographically. In the lexicographic ordering,  $x < y$  if for some  $i \in \{1, \dots, d\}$ ,  $x_j = y_j$ ,  $j < i$  and  $x_i < y_i$ . Then  $\ell(C(\omega))$  is the closest point in lexicographic and  $L^1$  distance from the origin such that  $\omega(\ell(C(\omega))) \in \mathcal{C}$ . The  $\ell(C(\omega))$  function is almost surely well-defined. By construction we have:

**Lemma 4.9** (Recoding lemma). *Given any non-trivial  $\mathcal{C} \subset \mathcal{S}$ , and the corresponding  $C(\omega)$  and  $\ell$  as above,*

$$\omega \mapsto \{\omega(\ell(C(T^z\omega)))\}_{z \in \mathbb{Z}^d}$$

*is a well-defined factor map on  $(\mathbf{A}^{\mathbb{Z}^d}, \mathcal{B}, \mathbb{P}, T)$ .*

The recoded space of arrow configurations is called  $(\Omega_{\mathcal{C}}, \mathcal{B}, \mathbb{P}, T)$  and has completely positive entropy. It has the additional property that it's enough to specify the arrow configuration and placement of bi-infinite trajectories in  $\mathcal{C}$  to completely determine  $\omega \in \Omega_{\mathcal{C}}$ .

*Proof of Theorem 2.12.* Assume that  $\mathcal{C}_\alpha = \{\omega \in \Omega_\alpha : h(\Omega_\alpha, \theta_\omega) = 0\}$  has positive measure. Since  $\theta_\omega$  is  $\mathcal{I}$  measurable,  $\mathcal{C}_\alpha$  is  $T_\alpha$  invariant.

For  $\epsilon > 0$ , using Theorem 4.8, choose length  $L_\alpha$  large enough, such that for any interval  $I \subset \mathbb{Z}$  longer than  $L_\alpha$ , there exists  $\mathcal{Y}_\alpha \subset \mathbf{A}^I$  with  $\mathbb{P}_\alpha(\mathcal{Y}_\alpha) \geq \mathbb{P}(\mathcal{C}) - \epsilon$  and

$$|\mathcal{Y}_\alpha| \leq e^{\epsilon|I|}. \quad (6)$$

Let  $\mathcal{C} = \pi_\alpha^{-1}(\mathcal{C}_\alpha) \subset \mathcal{S}$  be the set of arrow configurations such that the bi-infinite trajectory through the origin is in  $\mathcal{C}_\alpha$ . Let  $(\Omega_{\mathcal{C}}, \mathcal{B}, \mathbb{P}, T)$  be the factor obtained by applying the recoding lemma (Lemma 4.9) to  $\mathcal{C}$  and  $(\Omega, \mathcal{B}, \mathbb{P}, T)$ . The recoding leaves  $\mathcal{C}$  and  $\mathcal{C}_\alpha$  invariant, and in particular, the number of arrow configurations in  $\mathcal{C}_\alpha$  must grow sub-exponentially. Since the recoded system is

factor of the original, it must have positive entropy  $h > 0$ . Given  $\epsilon > 0$ , for all large enough rectangles  $R \subset \mathbb{Z}^d$ , Corollary 4.7 gives a set  $M \subset \mathbf{A}^R$  such that  $\mathbb{P}(\pi_R^{-1}(M)) \geq 1 - \epsilon$ , and

$$|M| \geq e^{(h-\epsilon)|R|}. \quad (7)$$

We recount the total number of configurations in  $\pi_R(\Omega_{\mathcal{C}})$  to show a contradiction. Let  $R$  have side-length  $L$ , and consider the cube  $R'$  with side-length  $3L$  that contains  $R$  in its center. Specifying the behavior of all bi-infinite trajectories in  $\mathcal{C}$  passing through the boundary of  $R'$  determines the bi-infinite trajectories in  $\mathcal{C}$  through  $R$ . For any other point in  $z \in R$ , the closest bi-infinite point in  $\ell$ -distance from  $z$  (see Lemma 4.9) must lie inside of  $R'$ , unless there are no bi-infinite points in  $R'$ . Let  $\mathcal{G}$  be the “bad” event that there exist no bi-infinite points in  $R'$ . By the ergodic theorem, there is an  $L$  so large such that the probability that there is no bi-infinite in  $R'$  is smaller than  $\epsilon$ .

There is another bad subset of  $\mathcal{C}$  that we must account for, where we cannot bound the number of bi-infinite trajectories using (6). Let  $\mathcal{Y} = \pi_{\alpha}^{-1}(\mathcal{Y}_{\alpha})$ , and let

$$b(\omega) = |\partial R' \cap (\mathcal{C} \setminus \mathcal{Y})|$$

be the number of points in  $\partial R'$  that are in  $\mathcal{C} \setminus \mathcal{Y}$ . Similarly define  $a(\omega) = |\partial R' \cap \mathcal{Y}|$ . If  $\mathcal{H} = \{b(\omega) \geq \sqrt{\epsilon}|\partial R'|\}$ , the Chebyshev inequality implies that  $\mathbb{P}(\mathcal{H}) \leq \sqrt{\epsilon}$ . Therefore, with high probability, there are not too many bad points on  $\partial R'$ . We partition the good set  $\Omega_{\mathcal{C}} \setminus \mathcal{H} \cup \mathcal{G}$  as

$$\mathcal{E}_j = \{b(\omega) = j\} \cap (\Omega_{\mathcal{C}} \setminus (\mathcal{H} \cup \mathcal{G})) \quad j = 0, \dots, \sqrt{\epsilon}|\partial R'|.$$

If  $z \in \partial R' \cap \mathcal{Y}$ , then on any bi-infinite trajectory of length  $3L$  beginning at  $z$ , there are at most  $e^{\epsilon 3L}$  possible arrow configurations. If  $z \in \partial R' \cap (\mathcal{C} \setminus \mathcal{Y})$ , there are  $d^{3L}$  arrow configurations. Then in  $\mathcal{E}_j$  there are at most

$$\binom{|\partial R'|}{j, a(\omega)} \exp(\epsilon(3L))^{a(\omega)} \exp(3L \log d)^j \leq 2^{|\partial R'|} \exp(C\sqrt{\epsilon}L^d)$$

sets corresponding to distinct arrow configurations in  $\mathbf{A}^R$ . Here, we've used  $a(\omega) \leq |\partial R'|$  and  $j \leq \sqrt{\epsilon}|\partial R'|$ . Therefore,

$$\begin{aligned} \sum_{j=0}^{\sqrt{\epsilon}|\partial R'|} |\{a \in \mathbf{A}^R : \mathbb{P}(\pi_R^{-1}(a) \cap \mathcal{E}_j) > 0\}| &\leq \sum_{j=0}^{\sqrt{\epsilon}|\partial R'|} 2^{|\partial R'|} \exp(C_1\sqrt{\epsilon}L^d) \\ &\leq \exp(C_2\sqrt{\epsilon}L^d). \end{aligned}$$

This contradicts (7) for small enough  $\epsilon$  and large enough  $L$ .  $\square$

## 5 Examples

We first prove Theorem 2.13 by creating walks with no asymptotic direction in  $d = 2$ . We build this example as a product space, so there exists  $S_1 : X \rightarrow X$

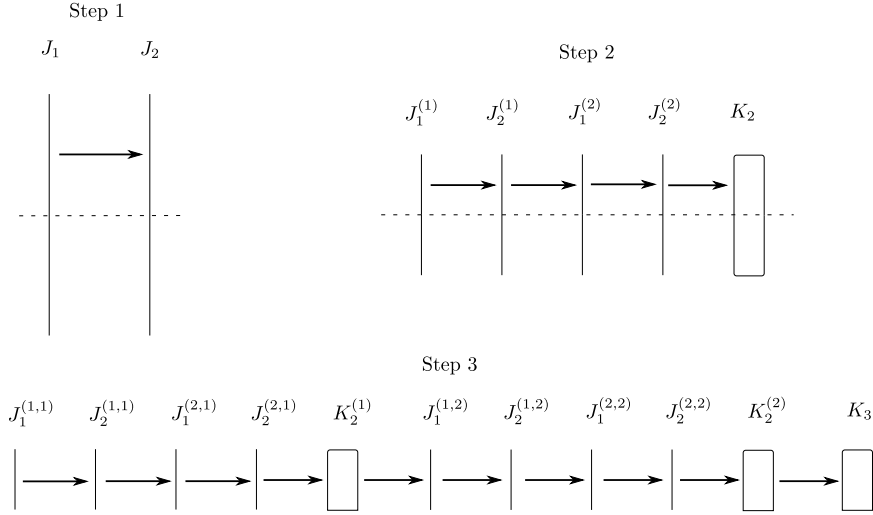


Figure 2: Three steps of the cutting and stacking construction of the intervals  $X$  and  $Y$ . The figure shows the case  $n_1 = n_2 = n_3 = 2$ . The dotted lines show where the intervals are cut. At each step, the intervals are cut, stacked horizontally, and a new “spacer”  $K_i$  is appended. The arrows show the mapping  $S_i$ ,  $i = 1, 2$ .

and  $S_2 : Y \rightarrow Y$  so that  $T^{(a,b)} : X \times Y \rightarrow X \times Y$  by  $T^{(a,b)}(x, y) = (S_1^a x, S_2^b y)$ .  $X$  and  $Y$  will be intervals in  $\mathbb{R}$  with Lebesgue measure.

We build  $(S_1, X)$  and  $(S_2, Y)$  by cutting and stacking construction. In fact they are Rank 1 and therefore ergodic [9].

**Building  $(S_1, X)$ :** Let  $n_1, \dots$  be a sequence of integers that are at least 2.  $X = [0, 1 + \sum_{i=2}^{\infty} \frac{1}{n_1 \cdots n_i})$ . We begin with  $n_1$  intervals of size  $\frac{1}{n_1}$ ,  $J_1, \dots, J_{n_1}$ . We define  $S_1(J_i) = J_{i+1}$  for  $i < n_1$ . Now subdivide each  $J_i$  into  $n_2$  intervals of size  $\frac{1}{n_1 n_2}$  called  $J_i^{(j)}$  and add an interval  $K_2$  of size  $\frac{1}{n_1 n_2}$ .  $S_1(J_i^{(j)}) = J_{i+1}^{(j)}$  for all  $i < n_1$ ,  $S_1(J_{n_1}^{(j)}) = J_1^{(j+1)}$  if  $j < n_2$  and lastly  $S_2(J_{n_1}^{(n_2)}) = K_2$ . Inductively at step  $k$  we have  $J_i^{(i_2, \dots, i_k)}, \dots, K_r^{(j_{r+1}, \dots, j_k)}, \dots, K_k$  of size  $\frac{1}{n_1 \cdots n_k}$ . We add another interval  $K_{k+1}$  of size  $\frac{1}{n_1 \cdots n_{k+1}}$  and subdivide the other intervals into  $n_{k+1}$  intervals of size  $\frac{1}{n_1 \cdots n_{k+1}}$ . Call these  $J_i^{(i_2, \dots, i_{k+1})}, \dots, K_r^{(j_{r+1}, \dots, j_{k+1})}, \dots, K_k^{(\ell)}$ . Let  $S_1(J_i^{(i_2, \dots, i_{k+1})}) = J_{i+1}^{(i_2, \dots, i_{k+1})}$  for  $i < n_1$ . Let  $S_1(J_{n_1}^{(i_2, \dots, i_{k+1})}) = J_1^{(i_2+1, \dots, i_{k+1})}$  if  $i_2 < n_2$ . Let  $S_1(J_{n_1}^{(n_2, \dots, i_{k+1})}) = K_2^{(i_3, \dots, i_{k+1})}$ . Let  $S_1(K_r^{(i_{r+1}, \dots, i_{k+1})}) = J_1^{(1, \dots, 1, i_{r+1}, \dots, i_{k+1})}$  unless  $i_{r+1} = n_{r+1}$  in which case  $S_1(K_r^{(n_{r+1}, i_{r+2}, \dots, i_{k+1})}) = K_{r+1}^{(i_{r+2}, \dots, i_{k+1})}$ .

**Lemma 5.1.**  $S_1^\ell(x) \in \cup_{i=r}^{\infty} K_i$  for some  $0 \leq \ell \leq (n_1 + 1) \cdots (n_r + 1)$ .

*Proof.* We prove this by induction on  $r$ . First we establish the base case of

$r = 2$ . Observe that if  $x \in \cup_{i=2}^{\infty} K_i$  then it is obviously true. Otherwise,  $x \in J_i^{(j)}$  and  $S_1^{n_1-i} x \in J_{n_1}^{(j)}$ . Now if  $j \leq n_2$  we have that  $S_1^{n_1}(J_{n_1}^{(j)}) = J_{n_1}^{(j+1)}$  and if  $j = n_2$  then  $S_1(J_{n_1}^{(j)}) = K_2$ . Applying this  $n_2 - j$  times we see  $S_1^{(n_2-j)n_1+1}(J_{n_1}^{(j)}) = K_2$ . Combining these, there exists  $r \leq n_1 - 1 + (n_2 - 1)n_1 + 1$  so that  $S_1^r(J_i^{(j)}) \subset \cup_{i=2}^{\infty} K_i$ .

The inductive step is similar. Assuming the result for  $K_\ell$  we prove it for  $K_{\ell+1}$ . So there exists  $a \leq (n_1 + 1) \cdots (n_\ell + 1)$  so that  $S_1^a(x) \in K_i$  for some  $i \geq \ell$ . If  $i > \ell$  we are done and so we assume that  $S_1^a(x) \in K_\ell^{(j)}$  for some  $j$ . Similar to before  $S^{(n_{\ell+1}-j)r} K_\ell^{(j)} \subset K_{\ell+1}$  where  $r \leq (n_1 + 1) \cdots (n_\ell + 1)$ . Indeed if  $j < n_\ell$  then  $S_1 K_\ell^{(j)} \subset J_1^{(1, \dots, 1, j+1)}$  and we may apply the inductive assumption to this point. Since  $(n_1 + 1) \cdots (n_\ell + 1)(n_{\ell+1}) + 1 \leq (n_1 + 1) \cdots (n_\ell + 1)(n_{\ell+1} + 1)$  we have the lemma.  $\square$

**Lemma 5.2.** *If  $j < n_{i+1}$  then  $S_1^r K_i^{(j)} \cap \cup_{\ell=i}^{\infty} K_\ell = \emptyset$  for all  $0 < r < n_1 \cdots n_i$ .*

This is similar to the proof of the previous lemma.

**Building  $(S_2, Y)$ :** This is similar. Let  $m_1, \dots$  be a sequence of integers that are at least 2. Let  $Y = [0, 1 + \sum_{i=2}^{\infty} \frac{1}{m_1 \cdots m_i})$ . As before we define intervals  $\hat{J}_\ell^{(i_2, \dots)}, \hat{K}_r^{(i_{r+1}, \dots)}$  and  $S_2$ . Analogously to before we have the following:

**Lemma 5.3.**  $S_2^\ell(x) \in \cup_{i=r}^{\infty} \hat{K}_i$  for some  $0 \leq \ell \leq (m_1 + 1) \cdots (m_i + 1)$ .

**Lemma 5.4.** *If  $j < m_{i+1}$  then  $S_2^r \hat{K}_i^{(j)} \cap \cup_{\ell=i}^{\infty} \hat{K}_\ell = \emptyset$  for all  $0 < r < m_1 \cdots m_i$ .*

For clarity we denote Lebesgue measure on  $X$  by  $\mu$  and Lebesgue measure on  $Y$  by  $\hat{\mu}$ .

**Definition 5.5.** *For any  $r > 0$ , define  $\alpha : X \times Y \rightarrow \{(1, 0), (0, 1)\}$  by*

$$\alpha(x, y) = \begin{cases} (1, 0) & \text{if } x \in \cup_{i=1}^{n_1} J_i \text{ or } x \in K_r \text{ and } y \in \cup \hat{J}_i \text{ or } \hat{K}_j \text{ with } j \leq r \\ (0, 1) & \text{else} \end{cases}$$

**Proposition 5.6.** *If  $\lim_{i \rightarrow \infty} \frac{n_1 \cdots n_i}{m_1 \cdots m_{i-1}} = \infty = \lim_{i \rightarrow \infty} \frac{m_1 \cdots m_i}{n_1 \cdots n_i}$  then almost every point of  $X \times Y$  defines a trajectory without an asymptotic direction.*

Since we cannot have a  $\mathbb{Z}^2$  system with bi-infinite trajectories that don't have asymptotic direction, this proves almost-sure coalescence and Theorem 2.13. We need a few lemmas to prove Prop. 5.6. Let

$$\hat{G}_r = \cup_{j=1}^{r-1} \cup_{\ell_j < m_j} \cup_{\ell_r=1}^{m_r(1-\frac{1}{r^2})-1} \cup_{i=1}^{m_1} \hat{J}_i^{(\ell_2, \dots, \ell_r)} \cup \hat{K}_2^{(\ell_3, \dots, \ell_r)} \cup \dots \cup \hat{K}_r^{(\ell_r)}.$$

That is,

$$\hat{G}_r^c = \cup_{i=m_r(1-r^{-2})}^{m_r} \hat{K}_r^{(i)} \cup_{j=r+1}^{\infty} \hat{K}_j$$

**Lemma 5.7.** *If  $r \geq 3$  and  $y \in \hat{G}_r$  then  $\alpha^{m_1 \cdots (\frac{m_r}{r^2}-1)}(x, y) = (a, b)$  where  $\frac{b}{a} \geq \frac{m_1 \cdots (m_r \frac{1}{r^2} - 1) - (n_1 + 1) \cdots (n_r + 1)}{(n_1 + 1) \cdots (n_r + 1)}$*

*Proof.* To see this, by Lemma 5.1,  $S_1^q(x) \in \cup_{i=r}^\infty K_i$  for some  $0 \leq q \leq (n_1 + 1) \cdots (n_r + 1)$ . So if  $q < j$  and  $S_2^j y \notin \cup_{\ell=r+1}^\infty \hat{K}_\ell$  for  $0 \leq i \leq j - q$  we have  $\alpha^j(x, y) = (0, 1)$ . Assuming  $y \in \hat{G}_r$  we will show that if  $i \leq m_1 \cdots (\frac{m_r}{r^2} - 1)$  then  $S_2^i y \notin \cup_{\ell=r+1}^\infty \hat{K}_\ell$ .

By our assumption that  $y \in \hat{G}_r$  we have the  $S_2^\ell y \in \hat{K}_r^{(j)}$  for  $j \leq m_r(1 - \frac{1}{r^2}) - 1$  where  $\ell \geq 0$  is minimal so that  $S_2^\ell y \in \hat{K}_r$ . Applying Lemma 5.4 we see that  $S_2^i(y) \notin \cup_{i=r+1}^\infty \hat{K}_i$  for all  $i < \ell + m_1 \cdots (\frac{m_r}{r^2})$ . The lemma follows.  $\square$

Similarly, let

$$G_r^c = \cup_{i=n_r(1-r^{-2})}^{n_r} K_r^{(i)} \cup_{j=r+1}^\infty K_j$$

**Lemma 5.8.** *If  $r \geq 3$  and  $x \in G_r$  then  $\alpha^{n_1 \cdots (\frac{n_r}{r^2} - 1)}(x, y) = (a, b)$  where  $\frac{a}{b} \geq \frac{n_1 \cdots (\frac{n_r}{r^2} - 1) - (m_1 + 1) \cdots (m_{r-1} + 1)}{(m_1 + 1) \cdots (m_{r-1} + 1)}$ .*

**Lemma 5.9.** *If  $m_i \geq i^2$  for all  $i$  we have  $\hat{\mu}(\hat{G}_r) \geq (1 - \frac{2}{r^2}) \frac{1}{1+2^{-r}} \mu(Y)$ .*

*Proof.* By construction  $\frac{\hat{\mu}(\hat{G}_r)}{\hat{\mu}(Y \setminus \cup_{\ell=r+1}^\infty \hat{K}_\ell)} = \frac{r^2 - 1}{r^2} - \frac{1}{m_r}$ . Since we are assuming  $m_i \geq 2$  for all  $i$  we have that

$$\hat{\mu}(Y) = \hat{\mu}(Y \setminus \cup_{\ell=r+1}^\infty \hat{K}_\ell) + \sum_{j=r+1}^\infty \hat{\mu}(\hat{K}_j) \leq \hat{\mu}(Y \setminus \cup_{\ell=r+1}^\infty \hat{K}_\ell) + \sum_{j=r+1}^\infty 2^{-j}.$$

So we obtain the lemma.  $\square$

The next lemma has an analogous proof.

**Lemma 5.10.** *If  $n_i \geq i^2$  for all  $i$  we have  $\mu(G_r) \geq (1 - \frac{2}{r^2}) \frac{1}{1+2^{-r}} \mu(X)$ .*

*Proof of Prop. 5.6.* First observe that by Lemmas 5.9 and 5.10

$$\cup_{i=1}^\infty \cap_{r=i}^\infty \hat{G}_r$$

and

$$\cup_{i=1}^\infty \cap_{r=i}^\infty G_r$$

have full measure. Next by the assumption of the proposition and Lemmas 5.7 and 5.8 we have that any such trajectory approximates both the vertical on an infinite sequence of times and the horizontal on an infinite sequence of times.  $\square$

*Proof of Theorem 2.13.* Proposition 5.6 proves that the trajectories have no asymptotic direction. We turn this into a geodesic walk by assigning weights to points on the lattice, and restricting the first-passage model to directed paths. If  $x \in S_1(\cup_{i=r}^\infty K_i)$  and  $y \notin \cup_{i=r+1}^\infty K_i$  let  $w(x, y) = 1$ . Similarly let  $w(x, y) = 1$  if  $y \in S_2(\cup_{i=r}^\infty \hat{K}_i)$  and  $x \notin \cup_{i=r}^\infty K_i$ . Otherwise let  $w(x, y) = \frac{1}{2}$ . Note that this gives an everywhere well defined map. Our walks in the previous proposition are geodesic walks for this set of weights because they only cross weights of  $\frac{1}{2}$  (and usually travel with a 1 either above or on their right).  $\square$

**Remark 5.11** (No asymptotic speed). This can be modified to give an example where the geodesics do not have an asymptotic speed. Let  $\hat{w}(x, y) = w(x, y)$  unless  $w(x, y) = \frac{1}{2}$  and  $x \in K_{2r}$  then  $\hat{w}(x, y) = \frac{3}{4}$ . Briefly, in Lemma 5.7 when  $r$  is even and the  $x$ -coordinate lands in  $K_r$  (as opposed to  $\cup_{i=r+1}^{\infty} K_i$ ) the walk will cross  $m_1 \cdots (\frac{m_r}{r^2} - 1)$  steps with weights of  $\frac{3}{4}$  in its first  $m_1 \cdots m_r$  steps. When  $r$  is odd there is a similar calculation with the weights being  $\frac{1}{2}$  instead.

*Proof of Corollary 2.15.* Let  $\mu$  be the measure from the previous system. Consider  $\mu^{\mathbb{Z}}$  on  $\mathbb{Z}^3$ . Observe that almost every trajectory remains in a 2-plane where it behaves as a trajectory from our previous model. So we do not have an asymptotic direction. However, almost every trajectory stays in a 2-plane and so we do not have almost sure coalescence.  $\square$

## 6 Appendix: Finite energy implies completely positive entropy

**Proposition 6.1.** *Finite energy implies every factor of the system (other than the one point system) has positive entropy. That is, the system has completely positive entropy.*

This is straightforward and probably known, though we could not find a reference. The entropy of a partition  $K = \{K_1, \dots, K_r\}$  is defined as usual by

$$h(\Omega, S, K) = \lim_{n \rightarrow \infty} -\frac{1}{(2n-1)^d} \sum_{A \in \bigvee_{i=0}^n S^i K} \log \mathbb{P}(A) \mathbb{P}(A).$$

*Proof.* We prove the contrapositive. Without loss of generality, let  $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ , and assume  $(\Omega, \mathcal{F}, S, \mu)$  has a nontrivial measurable factor with zero entropy. Then there exists a non-trivial partition  $\mathcal{K} = \{K_1, \dots, K_r\}$  a finite nontrivial measurable partition with  $h(\Omega, S, \mathcal{K}) = 0$ .

Let  $\hat{C}_N(\vec{x}) = \{\vec{y} : y_{i_1, \dots, i_d} = x_{i_1, \dots, i_d} \text{ for all } 0 < |i_j| \leq N\}$ . Let  $\hat{A}_N(\vec{x}) = \{\vec{y} \in \hat{C}_N(\vec{x}) : \vec{x}_0 = \vec{y}_0\}$  and  $\hat{B}_N(\vec{x}) = \{\vec{y} \in \hat{C}_N(\vec{x}) : \vec{x}_0 \neq \vec{y}_0\}$ .

**Lemma 6.2.** *It suffices to show that for any  $\epsilon > 0$  there exists  $N$  so that there exists a finite set  $F \subset \{0, 1\}^{\mathbb{Z}^d}$  so that  $\mu(\cup_{\vec{x} \in F} \hat{A}_N(\vec{x})) > 1 - \epsilon$  and  $\mu(\cup_{\vec{x} \in F} \hat{B}_N(\vec{x})) < \epsilon$ .*

*Proof.* Choose  $\epsilon_i = 2^{-i}$  and corresponding  $N_i$  and  $F_i$ . Let  $G = \cap_{N=1}^{\infty} \cup_{i=N}^{\infty} (\cup_{\vec{x} \in F_i} \hat{A}_i(\vec{x}))$  we have that  $\mu(\cup_{\vec{z} \in G} \hat{B}_{\infty}(\vec{z})) = 0$ . Here,  $\hat{B}_{\infty}(\vec{x})$  is defined as the monotone limit  $\lim_{N \rightarrow \infty} \hat{B}_N(\vec{x})$ . Without loss of generality, we assume  $\{\vec{x} \in G : \vec{x}_0 = 0\}$ . Considering coloring  $\vec{0}$  to be 1 on  $\{\vec{x} \in G : \vec{x}_0 = 0\}$  we see we can not have finite energy.  $\square$

**Lemma 6.3.** *Let*

$$\tilde{A}_N(\vec{x}) = \{\vec{y} : S^{i_1, \dots, i_d}(\vec{y}) \in K_i \text{ iff } S^{i_1, \dots, i_d}(\vec{x}) \in K_i \text{ for all } |i_j| \leq N\}$$

and

$$\tilde{B}_N(\vec{x}) = \{\vec{y} : S^{i_1, \dots, i_d}(\vec{y}) \in K_i \text{ iff } S^{i_1, \dots, i_d}(\vec{x}) \in K_i \text{ for all } 0 < |i_j| \leq N, \\ \text{and } \vec{x} \in K_\ell \text{ implies } \vec{y} \in K_\ell^c\}.$$

If  $h(\Omega, S, \mathcal{K}) = 0$  then for all  $\epsilon > 0$  there exists  $N$  and a finite set  $F$  so that  $\mu(\cup_{\vec{x} \in F} \tilde{A}_N(\vec{x})) > 1 - \epsilon$  and  $\mu(\cup_{\vec{x} \in F} \tilde{B}(\vec{x})) < \epsilon$ .

If not then the entropy of the partition is positive.

**Lemma 6.4.** For any  $N \in \mathbb{N}$  and  $\epsilon > 0$  there exists  $M$  so that if  $\mathcal{C}$  is an  $N$  cylinder and a finite set of  $M$  cylinders  $C_1, \dots, C_n$  so that  $\mu(\mathcal{C} \Delta \cup_{j=1}^n C_j) < \mu(\mathcal{C})\epsilon$ .

This is a straightforward.

Let  $R$  be the  $N$  in Lemma 6.3 and  $\hat{F}$  be the corresponding  $F$ . Let  $M$  be the  $M$  in Lemma 6.4 for  $N = R$ . For each  $\vec{x} \in \hat{F}$  let  $C_1^{\vec{x}}, \dots, C_{n_{\vec{x}}}^{\vec{x}}$  be so that  $\mu(\mathcal{C} \Delta \cup_{i=1}^{n_{\vec{x}}} C_i^{\vec{x}}) < \epsilon \mu(\mathcal{C})$  where  $\mathcal{C}$  is the  $R$  cylinder containing  $\vec{x}$ . For each  $C_i^{\vec{x}}$  let  $\vec{y}_i^{\vec{x}}$  be a point in the cylinder. Consider  $N = R + M$ , and let  $F$  be the finite set  $\{\vec{y}_i^{\vec{x}}\}_{\vec{x} \in \hat{F}, i \leq n_{\vec{x}}}$ . We verify the sufficient condition for Lemma 6.2 with  $N = R + M$  and  $F$  as above and  $\epsilon = \epsilon + 2\epsilon$ . Indeed

$$\mu\left(\cup_{\vec{x} \in \hat{F}} \cup_{i=1}^{n_{\vec{x}}} \hat{A}_R(\vec{y}_i^{\vec{x}})\right) \geq \mu\left(\cup_{\vec{x} \in \hat{F}} \tilde{A}_R(\vec{x}) - \epsilon\right), \\ \mu\left(\cup_{\vec{x} \in \hat{F}} \cup_{i=1}^{n_{\vec{x}}} \hat{B}_R(\vec{y}_i^{\vec{x}})\right) \leq \mu\left(\cup_{\vec{x} \in \hat{F}} \tilde{B}_R(\vec{x}) + \epsilon\right).$$

□

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